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Evaluation of eigenfunctions from compound matrix variables in non-linear elasticity – I. Fourth order systems

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Abstract

We show how the compound matrix method can be used to produce eigenfunctions as well as eigenvalues for bifurcation problems in non-linear elasticity. For typical problems in elasticity the boundary conditions require a different treatment to that required for typical problems in fluid mechanics. For elasticity problems we have to use an additional shooting method to ensure that the boundary conditions are satisfied.

Keywords: Compound matrices; Eigenfunctions; Stability; Bifurcation; Non-linear elasticity; 73C; 73G; 73H

1. Introduction

The compound matrix method introduced by Ng and Reid [1] has proved very successful for determining critical parameters (eigenvalues) for homogeneous problems in fluid mechanics, in particular for stiff problems such as the Orr–Sommerfeld equations. The related problem of determining the corresponding eigenfunctions from the compound matrix variables has been addressed by Ng and Reid [2] and also by Straughan and Walker [3]. In both [2,3] a fourth order problem was considered. In [2] the system was assumed to be formulated as a single fourth order equation while in [3] the more natural assumption of two second order equations was used. However, both approaches are essentially equivalent. To find the eigenfunction the process given in [2,3] is as follows. Once the critical parameter (eigenvalue) has been found, a new differential equation is proposed whose coefficients are the previously determined compound matrix variables. It is then proved, via an identity, that the solution to this new equation is also a solution to the original problem. Attention is then focused on the boundary conditions for the original problem and this raises some difficulties due to incompatibilities of the boundary conditions and the asymptotic solution to the new eigenfunction equation. These difficulties are resolved by integrating the new eigenfunction equation inwards towards the boundary layer. The asymptotic form of the new eigenfunction equation now guarantees that the boundary conditions for the

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original problem will be satisfied automatically. Unfortunately this method does not apply to problems in elasticity due to the quite different nature of the typical boundary conditions.

Bifurcation problems in elasticity require a similar numerical approach. There are many problems of interest where the incremental partial differential equations can be reduced to a fourth order ordinary differential system through the use of symmetry and separation of variables, for example. Normally stiffness is not a problem and a simple determinantal method such as that proposed in [4] is adequate. Although the numerical values of the determinant involved can be very large, particularly for higher order problems. In any event, the ease of implementation and robustness of the compound matrix method has led to it being the method of choice for more recent problems [5]. For many problems we only require the critical parameter (often referred to as the eigenvalue) for the problem. However, for some problems the corresponding eigenfunction is required as a fundamental part of the overall solution, see [6], for example, which we will consider in more detail below and which provided the motivation for this work. We then have the problem of determining the corresponding eigenfunction. The main differences when compared to typical fluid mechanics problems, as typified in [2,3], are that the coefficients of the original differential equations and the boundary conditions are more complicated as they contain combinations of instantaneous moduli (and possibly their derivatives) which, in turn, contain the desired critical parameter (which may typically be a loading or deformation parameter, a geometric parameter or perhaps a material parameter from the strain–energy function) in some complicated manner. We find that we have to do some additional work to ensure that the boundary conditions of the original problem are satisfied.

In this paper we consider what might be regarded as a typical problem from non-linear elasticity. In Section 2 we describe the basic compound matrix method for such a system of two simultaneous second order ordinary differential equations. We then apply the eigenfunction method described in [2,3] and give a direct proof that the solution to the new second order eigenfunction equation is also a solution of the original problem. This is an alternative proof to that given by Straughan and Walker [3]. We then consider the boundary conditions and find that we have to develop a new method of solution. The method that we formulate requires an additional shooting method when compared to the typical fluid problem. We show through examples that our new compound matrix method works very well.

In the examples that we consider below we will compare results from three different methods. The naive determinantal method, an exact solution (which still requires numerical evaluation in the non-trivial example) and the compound matrix method. There are other methods that we might consider. In particular the Chebyshev tau method [3]. However, this will not be particularly easy to implement for elasticity problems due to the nature of the coefficients in both the differential equations and the boundary conditions. A modification of the determinantal method has been proposed in [7]. This method avoids the numerical evaluation of a determinant but requires the symbolic differentiation of one. For typical problems in elasticity this will require the taking derivatives of the coefficients of the boundary conditions. This can involve a significant amount of work and so we do not consider it here.

2. Compound matrix method

Here we consider two second order equations for $f(x)$, $h(x)$ in the form

$$f'' = \alpha_1 f + \alpha_2 f' + \alpha_3 h + \alpha_4 h', \tag{1}$$

$$h'' = \gamma_1 f + \gamma_2 f' + \gamma_3 h + \gamma_4 h', \tag{2}$$

where the prime denotes differentiation with respect to x and the coefficients α_i, γ_i, $i = 1 \ldots 4$, will depend on the parameter λ, say, that we are looking for and, in general, on x. We also impose the boundary conditions

$$a_1 f' + a_2 f + a_3 h = 0, \quad x = a, \tag{3}$$

$$b_1 h' + b_2 f + b_3 h = 0, \quad x = a \tag{4}$$

and

$$c_1 f' + c_2 f + c_3 h = 0, \quad x = b, \tag{5}$$

$$d_1 h' + d_2 f + d_3 h = 0, \quad x = b. \tag{6}$$

The coefficients in the boundary conditions will also depend on the parameter λ, as will the ends of the range of integration a and b. Typically for problems in elasticity we have the same two boundary conditions to be applied at both end of the range and so $c_1 = a_1$ etc. We suppose that the Eqs. (1) and (2) are solved, in principle, with two linearly independent initial conditions (at $x = a$) which ensure that the boundary conditions (3) and (4) are satisfied. The two solutions thus obtained are labelled f^1, f^2, h^1 and h^2. The full solution can then be written

$$f(x) = C_1 f^1(x) + C_2 f^2(x) \tag{7}$$

and

$$h(x) = C_1 h^1(x) + C_2 h^2(x), \tag{8}$$

where C_1 and C_2 arbitrary constants. We now introduce six new compound matrix variables $\phi_i(x)$, defined by 2×2 determinants

$$\begin{aligned}
\phi_1 &= \begin{vmatrix} f^1 & f^2 \\ f^{1'} & f^{2'} \end{vmatrix}, \quad \phi_2 = \begin{vmatrix} f^1 & f^2 \\ h^1 & h^2 \end{vmatrix}, \quad \phi_3 = \begin{vmatrix} f^1 & f^2 \\ h^{1'} & h^{2'} \end{vmatrix}, \\
\phi_4 &= \begin{vmatrix} f^{1'} & f^{2'} \\ h^1 & h^2 \end{vmatrix}, \quad \phi_5 = \begin{vmatrix} f^{1'} & f^{2'} \\ h^{1'} & h^{2'} \end{vmatrix}, \quad \phi_6 = \begin{vmatrix} h^1 & h^2 \\ h^{1'} & h^{2'} \end{vmatrix}.
\end{aligned} \tag{9}$$

Next we differentiate (9) and use (1) and (2) with f and h replaced with f^1, etc., as required, to obtain the compound matrix differential equations

$$\begin{aligned}
\phi_1' &= \alpha_2\phi_1 + \alpha_3\phi_2 + \alpha_4\phi_3, \\
\phi_2' &= \phi_3 + \phi_4, \\
\phi_3' &= \phi_5 + \gamma_2\phi_1 + \gamma_3\phi_2 + \gamma_4\phi_3, \\
\phi_4' &= \phi_5 + \alpha_1\phi_2 + \alpha_2\phi_4 - \alpha_4\phi_6, \\
\phi_5' &= \alpha_1\phi_3 + \alpha_2\phi_5 + \alpha_3\phi_6 - \gamma_1\phi_1 + \gamma_3\phi_4 + \gamma_4\phi_5, \\
\phi_6' &= -\gamma_1\phi_2 - \gamma_2\phi_4 + \gamma_4\phi_6.
\end{aligned} \tag{10}$$

Now using (3), (4), arbitrarily normalising $\phi_2(a) = 1$ and assuming that $a_1(a) \neq 0$ and $b_1(a) \neq 0$ we have the initial conditions at $x = a$

$$\begin{aligned}
\phi_1 &= -a_3/a_1, \\
\phi_2 &= 1, \\
\phi_3 &= -b_3/b_1, \\
\phi_4 &= -a_2/a_1, \\
\phi_5 &= (a_2 b_3 - a_3 b_2)/a_1 b_1, \\
\phi_6 &= b_2/b_1.
\end{aligned} \tag{11}$$

We note that when a_1 and or b_1 is zero at $x = a$ we can still find suitable initial conditions without any difficulty. Typically the initial conditions for problems in fluids would be $f(a) = h(a) = 0$ which leads to $\phi_i = 0$ except for ϕ_5 which would be set to unity, see Straughan and Walker [3], for example.

It remains to ensure that the boundary conditions at $x = b$ are satisfied. We use the solution (7) and (8) and the derivatives of these equations

$$f' = C_1 f^{1'} + C_2 f^{2'} \tag{12}$$

and

$$h' = C_1 h^{1'} + C_2 h^{2'}. \tag{13}$$

If we substitute (7), (8) and (12) and (13) into the boundary conditions (5 and 6) we have

$$\begin{bmatrix} c_1 f^{1'} + c_2 f^1 + c_3 h^1 & c_1 f^{2'} + c_2 f^2 + c_3 h^2 \\ d_1 h^{1'} + d_2 f^1 + d_3 h^1 & d_1 h^{2'} + d_2 f^2 + d_3 h^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0, \quad x = b. \tag{14}$$

We then require the 2×2 determinant of the coefficient matrix in (14) to be zero for the existence of non-trivial solutions. The 2×2 determinant can be written in terms of ϕ_i and setting this to be zero gives us our target condition

$$c_1(d_1\phi_5 - d_2\phi_1 + d_3\phi_4) + c_2(d_1\phi_3 + d_3\phi_2) + c_3(d_1\phi_6 - d_2\phi_2) = 0, \quad x = b. \tag{15}$$

For comparison, we note that the target condition for the fluid problem considered by Straughan and Walker [3] was $\phi_2(b) = 0$. At this point we need to adjust the value of our parameter λ to ensure that (15) is satisfied. This is accomplished by using the usual shooting method.

3. Compound matrix eigenfunction

Now suppose that we have found a critical value of our parameter λ so that (10) with (11) integrate to give (15). We can then arrange to obtain values of $\phi_i(x)$ for any $x \in [a, b]$. The method for determining the corresponding eigenfunctions starts by formally solving (7) and (8) for the constants C_1 and C_2 and then substituting these values into (12) and (13). We then have

$$\phi_2 f' = \phi_4 f + \phi_1 h \tag{16}$$

and

$$\phi_2 h' = -\phi_6 f + \phi_3 h, \tag{17}$$

having multiplied by ϕ_2 which we assume is non-zero throughout the range $x \in (a, b)$. We recall that we arbitrarily set $\phi_2(a) = 1$. If $\phi_2 = 0$ at some point $x = x_1$, say, then the constants C_1 and C_2 in (7) and (8) would not exist and so we would not have a solution for f and h. Eqs. (16) and (17) are then the eigenfunction equations that we use to determine f and h.

If we consider (16) and (17) at $x = a$ we can substitute the initial conditions for the ϕ_i's from (11) and we see that the initial conditions (3), (4) for the original problem are satisfied. Thus we are free to impose any initial conditions on $f(a)$ and $h(a)$. To normalise the solution we set

$$f(a) = 1.$$

As we shall see below we must choose a particular value for $h(a)$ (h_a say) in order that the remaining boundary conditions (5) and (6) are satisfied.

Firstly we prove that a solution to (16) and (17) with initial conditions $f(a) = 1$ and $h(a) = h_a$ is also a solution to the original problem (1) and (2). This is an alternative, more direct, proof to that given by Straughan and Walker [3] (which, in turn, used the results given in [2]). We shall focus attention on the equation for f but, as will be obvious, the same argument can be used for the other equation. First we differentiate (16) to obtain

$$\phi_2 f'' + \phi_2' f' = \phi_4' f + \phi_1' h + \phi_4 f' + \phi_1 h'. \tag{18}$$

Now use Eq. (10) to substitute for the ϕ_i derivatives so that

$$\phi_2 f'' = -\phi_3 f' + (\phi_5 + \alpha_1\phi_2 + \alpha_2\phi_4 - \alpha_4\phi_6) f + (\alpha_2\phi_1 + \alpha_3\phi_2 + \alpha_4\phi_3) h + \phi_1 h'. \tag{19}$$

Next we subtract ϕ_2 times the right hand side of (1) from both sides of (19) to get

$$\phi_2 L_1(f, h) = -(\phi_3 + \alpha_2\phi_2) f' + (\phi_5 + \alpha_2\phi_4 - \alpha_4\phi_6) f + (\alpha_2\phi_1 + \alpha_4\phi_3) h + (\phi_1 - \alpha_4\phi_2) h', \tag{20}$$

where $L_1(f, h)$ is the differential equation (1). We now use (16) and (17) to substitute for f' and h' so that we may write

$$\phi_2^2 L_1(f, h) = (\phi_2\phi_5 - \phi_1\phi_6 - \phi_3\phi_4) f. \tag{21}$$

We now recognise from (9) that we have an identity

$$\phi_2\phi_5 - \phi_1\phi_6 - \phi_3\phi_4 \equiv 0, \tag{22}$$

as pointed out by Ng and Reid [2]. Hence Eq. (1) holds. Similarly, using the same identity, for (2).

We shall now assume that the initial condition $h(a) = h_a$ has been chosen so that the boundary condition (5) is satisfied. It remains to be shown that the remaining boundary condition (6) is also satisfied. To do this we first consider (5) and write, using (16)

$$\begin{aligned}\phi_2(c_1f' + c_2f + c_3h) &= c_1(\phi_4f + \phi_1h) + \phi_2(c_2f + c_3h),\\ &= (c_1\phi_4 + c_2\phi_2)f + (c_1\phi_1 + c_3\phi_2)h = 0, \quad x = b.\end{aligned} \tag{23}$$

For the left hand side of the remaining boundary condition (6) we have

$$\phi_2(d_1h' + d_2f + d_3h) = (d_2\phi_2 - d_1\phi_6)f + (d_1\phi_3 + d_3\phi_2)h. \tag{24}$$

Now use (23) to substitute for h in the right hand side of (24) so that

$$\phi_2(d_1h' + d_2f + d_3h) = \frac{(d_2\phi_2 - d_1\phi_6)(c_1\phi_1 + c_3\phi_2) - (d_1\phi_3 + d_3\phi_2)(c_1\phi_4 + c_2\phi_2)}{(c_1\phi_1 + c_3\phi_2)}f, \tag{25}$$

or, by rearranging

$$\begin{aligned}\phi_2(c_1\phi_1 + c_3\phi_2)(d_1h' + d_2f + d_3h) = \{&c_1(d_2\phi_1\phi_2 - d_1(\phi_1\phi_6 + \phi_3\phi_4) - d_3\phi_2\phi_4) - c_2\phi_2(d_1\phi_3 + d_3\phi_2)\\ &+ c_3\phi_2(d_2\phi_2 - d_1\phi_6)\}f.\end{aligned} \tag{26}$$

Using the identity (22) this becomes

$$(c_1\phi_1 + c_3\phi_2)(d_1h' + d_2f + d_3h) = \{c_1(-d_1\phi_5 + d_2\phi_1 - d_3\phi_4) - c_2(d_1\phi_3 + d_3\phi_2) + c_3(d_2\phi_2 - d_1\phi_6)\}f. \tag{27}$$

We see from the target condition (15) that the right hand side of (27) is identically zero and so the second boundary condition is satisfied, provided that $(c_1\phi_1 + c_3\phi_2)$ is not zero at $x = b$.

To summarise, the method for determining f and h is to normalise the solution by setting $f(a) = 1$ then choose a value for $h(a) = h_a$, integrate Eqs. (16) and (17) to $x = b$ and then adjust the value of h_a to ensure that one of (5) or (6) is satisfied. We may, of course, choose some equivalent combination of the two boundary conditions. For the examples considered below we simply repeat the calculation twice, successively using one of the boundary conditions. We then calculate the residuals produced by the two boundary conditions and take the solution which produces the smallest residual. In practice the difference between the two residuals is small.

4. Examples

The purpose of this section is first to verify that the compound matrix method does work and secondly to compare the results it gives with one other possible approach.

4.1. A simple artificial example

As a first trivial example we consider the problem given by

$$f'' = (h - f)/2, \tag{28}$$

$$h'' = (f - h)/2, \tag{29}$$

with

$$f' + h = 0, \quad x = 0, \tag{30}$$

$$h' + f = 0, \quad x = 0 \tag{31}$$

and

$$f' + h = 0, \quad x = \pi, \tag{32}$$

$$h' + \lambda f = 0, \quad x = \pi. \tag{33}$$

It is easy to show that the solution is zero unless the parameter $\lambda = 1$. In this case the solution normalised so that $f(0) = 1$ is given by

$$f = \sin x + \cos x, \quad h = -(\sin x + \cos x). \tag{34}$$

In Table 1. we give the errors in calculating first λ and then $f(\pi)$, $h(0)$ and $h(\pi)$ using two methods. Firstly, the determinantal method. Briefly, this method starts with the solution for $f(x)$ and $h(x)$ in the form (7) and (8). For this method the functions f^i, h^i, $i = 1, 2$, are obtained numerically by integrating (1) and (2) twice. Each time we regard the system as an initial value problem with two linearly independent initial conditions which incorporate the boundary conditions (3) and (4). We substitute these two solutions into the two remaining boundary conditions (5) and (6). This gives two homogeneous equations for the constants C_1 and C_2. For non-trivial solutions we then set the appropriate 2×2 determinant to be zero. This determines the critical value for λ. With this critical value for λ we compute the 2×2 coefficient matrix and determine its eigenvalues and eigenvectors. At least one eigenvalue will be close to zero, we take the smallest eigenvalue and the corresponding eigenvector to give C_1, C_2 and hence $f(x)$ and $h(x)$ from (7) and (8).

The results in Table 1 clearly show that both methods can obtain reasonable results. It is difficult to make a direct comparison between the two methods as they are quite different. For example, the routine to find the eigenvectors for the determinantal method just uses machine accuracy while the differential equation solvers in the compound matrix method require a specified tolerance. Asking for consistently more accurate results can improve the compound matrix results at the expense of speed but there is a limit when the differential equation routines start to have difficulty in achieving the required accuracy. Here we are using a simple fourth order Runge Kutter Fehlberg method. There is some scope for using more accurate methods to improve the results.

4.2. A tube under axial compression

In [6] we looked at the problem of a circular tube that was compressed between "greased end plates". The tube will deform into a shorter circular tube until some critical deformation is reached when it may buckle into one or more distinct modes. (For longer tubes we recover the familiar Euler strut flexural mode but there are other possibilities). For this example we choose a number of physical parameters so that we only have to consider the axial stretch which we will call λ =(original length)/(deformed length) of the tube with $0 < \lambda < 1$.

The tube is composed of an unconstrained elastic material with strain–energy function

$$W = \frac{\mu}{2}(I_1 - 3) - (\kappa + \mu/3)\log(J) - (2\mu/3 - \kappa)(J - 1), \tag{35}$$

where $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $J = \lambda_1\lambda_2\lambda_3$ and λ_i, $i = 1, 2, 3$ are the principal stretches and, for this problem, $\lambda = \lambda_3$. Here we normalise the strain–energy by taking the shear modulus $\mu = 1$ and we consider a highly compressible material with a bulk modulus $\kappa = 5$. The cylindrical tube has a non-dimensionalised undeformed outer radius $B = 1$ and an undeformed inner radius $A = 2/3$. The undeformed length of the short tube (disc) is taken to be $\pi/7$.

With these choices of parameters the incremental equations have the explicit form

$$(96\lambda + 208\lambda^2)r^2 f'' + (96\lambda + 208\lambda^2)rf' + ((-1911\lambda^2 - 441\lambda)r^2 - 208\lambda^2 - 96\lambda)f + (1456\lambda + 336)r^2 h' \tag{36}$$

Table 1
Errors in the determinantal and compound matrix methods

	Determinantal	Compound matrices
λ	4.2×10^{-8}	1.1×10^{-7}
$f(\pi)$	8.5×10^{-8}	1.7×10^{-7}
$h(0)$	1.3×10^{-7}	4.3×10^{-9}
$h(\pi)$	1.1×10^{-7}	1.6×10^{-7}

and

$$48h''\lambda^2 r + (-1456\lambda^2 - 336\lambda)rf' + (-1456\lambda^2 - 336\lambda)f + 48h'\lambda^2 + (-2352 - 10192\lambda - 441\lambda^2 - 1911\lambda^3)hr, \tag{37}$$

where $f(r)$ and $h(r)$ are the incremental displacements in the radial and axial directions and r is the deformed radial coordinate. The exact solution can be written

$$f(r) = C_1 I_1\left(\frac{7k_1 r}{\lambda}\right) + C_2 I_1\left(\frac{7k_2 r}{\lambda}\right) + C_3 K_1\left(\frac{7k_1 r}{\lambda}\right) + C_4 K_1\left(\frac{7k_2 r}{\lambda}\right) f(r) \tag{38}$$

and

$$\begin{aligned}16k_1k_2(3+13\lambda)h(r) = {} & (-16(6+13\lambda)k_1^2 + 3\lambda^2(3+13\lambda))I_0\left(\frac{7k_1 r}{\lambda}\right)C_1 + (-16(6+13\lambda)k_2^2 \\ & + 3\lambda^2(3+13\lambda))I_0\left(\frac{7k_2 r}{\lambda}\right)C_2 + (16(6+13\lambda)k_1^2 - 3\lambda^2(3+13\lambda))K_0\left(\frac{7k_1 r}{\lambda}\right)C_3 \\ & + (16(6+13\lambda)k_2^2 - 3\lambda^2(3+13\lambda))K_0\left(\frac{7k_2 r}{\lambda}\right)C_4,\end{aligned} \tag{39}$$

where C_1, C_2, C_3 and C_4 are constants

$$\begin{aligned} k_1 &= \frac{\lambda}{4}\sqrt{3+13\lambda}, \\ k_2 &= \frac{1}{4}\sqrt{\frac{9\lambda^2 + 39\lambda^3 + 48 + 208\lambda}{6+13\lambda}} \end{aligned} \tag{40}$$

and I_0, I_1, K_0 and K_1 are modified Bessel functions of the first and second kind. The boundary conditions are

$$(6+13\lambda)rf' + 13\lambda f + 91rh, \quad r = a, b \tag{41}$$

and

$$\lambda h' - 7f, \quad r = a, b, \tag{42}$$

where $a = 8/3\sqrt{3+13\lambda}$ and $b = 4/\sqrt{3+13\lambda}$.

Although this problem admits an exact solution in terms of Bessel functions we still have to evaluate the zero's of a 4×4 determinant in order to find the bifurcation parameter λ. Having found λ we compute the 4×4 matrix and find the eigenvector corresponding to the smallest eigenvalue (which, ideally, will be zero) for the constants C_i. In this case the numerical methods used for the evaluation of the eigenvalues and eigenvectors give an imaginary part that is identically zero. This is an exceptional case and normally both eigenvalues and eigenvectors could be expected to have small imaginary parts. Given that the problem is purely real any such imaginary components of the solution can be ignored and regarded as an unavoidable numerical error.

Since we do not have a simple numerical evaluation of the exact results we cannot be sure that they will be anymore accurate than either of the two other methods that we consider. For this reason we simply present the real parts of the solutions in Table 2. For the results presented in Table 2 we have set the tolerances for the differential equation solvers in the compound matrix method to be small enough for a relative error of 5×10^{-7}. There are two distinct solutions for the parameter λ and we give both of these. As we can see from Table 2 all three methods give very similar results for the first bifurcation point with the exact solution and the compound matrix method giving marginally closer results than the determinantal method. This is reversed at the second bifurcation point where the compound matrix method gives slightly different results at the end of the range, that is, for $f(b)$ and $h(b)$. However, if we decrease the tolerance in all of the differential equation solvers the compound matrix solutions do move towards the other two solutions. For example with a relative error tolerance set at $= 1 \times 10^{-10}$ then $f(b)/f(a) = -0.6981119$ and $h(b)/f(a) = -0.4171635$ and all of the results agree to five significant figures.

Table 2
Solutions obtained from the exact solution, the compound matrix method and the determinantal method: rounded to seven figures

	Exact	Compound matrices	Determinantal
λ	0.5980753	0.5980754	0.5980753
$f(b)$	0.9258547	0.9258547	0.9258545
$h(a)$	−0.7286607	−0.7286607	−0.7286607
$h(b)$	0.5754328	0.5754328	0.5754326
Second root			
λ	0.3842477	0.3842477	0.3842477
$f(b)$	−0.6981094	−0.6981428	−0.6981098
$h(a)$	−0.6599870	−0.6599868	−0.6599870
$h(b)$	−0.4171616	−0.4171754	−0.4171618

5. Concluding remarks

We have established that the compound matrix method for eigenfunctions proposed in [2,3] for fourth order problems in fluid mechanics can be developed to deal with similar problems in elasticity. The new feature of the method is the approach to the boundary conditions.

The two methods we have looked at for elasticity problems both produce satisfactory results. It is surprising how well the determinantal method does, particularly for the second bifurcation point where we are looking at extreme deformations ($\lambda \simeq 0.4$ represents a cylinder that has been reduced to only 40% of its original height). This is due to the fact that we only have to deal with a 2×2 matrix. Many current problems of interest within elasticity involve multi-component systems, an artery, for example, is modelled by a three layer system. Analysis of such systems can lead to differential systems of 12th order for the axisymmetric case and a 6×6 matrix. We might reasonable expect the accuracy and reliability of the determinantal method to be inversely proportional to the order of the system. In contrast the compound matrix method should simply scale up with no loss of accuracy, at least for the eigenvalue problem. However, it is not immediately apparent how the compound matrix eigenfunction method can be applied to higher order systems. This is the topic of a companion paper.

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